Uniformization of log Fano pairs

Y smooth complex manifold
$G \curvearrowright Y$ discrete properly freely
$X = Y/G$ smooth manifold.
Given X, (Y, G) ? $(\tilde{X}, \pi_1(X))$

Thm [Poincaré–Koebe 1907]:
X Riemann surface compact connected
g its genus

$$\tilde{X} = \begin{cases} \mathbb{P}^1(\mathbb{C}) & \text{if } g = 0 \\ \mathbb{C} & = 1 \\ \mathbb{D} & \geq 2 \end{cases}$$

Higher dimensions? (X, ω)

Thm [Hawley–Igusa, 53-54]
(X,ω) compact Kähler, $(\tilde{X}, \tilde{\omega})$ $\pi^*\omega =: \tilde{\omega}$
$\pi : \tilde{X} \longrightarrow X$ universal cover. The holomorphic sect. curvature of (X,ω) is constant $\lambda \in \mathbb{R}$, $\dim X = n$

$$(\tilde{X}, \tilde{\omega}) \underset{\text{hol. isom.}}{\simeq} \begin{cases} (\mathbb{P}^n, \frac{1}{\lambda}\omega_{FS}) & \lambda > 0 \\ (\mathbb{C}^n, \omega_{euc}) & \lambda = 0 \\ (\mathbb{B}^n, \frac{-1}{\lambda}\omega_{Bergman}) & \lambda < 0 \end{cases}$$

Rks:
- Complex analoguous to Killing–Hopf theorem
- If $(\tilde{X}, \tilde{\omega}) \simeq (\mathbb{P}^n, \omega_{FS}) \simeq (X, \omega)$
  If $\pi_1(X) \neq 1$, $g \in \pi_1(X) - \{1\}$, $\leqslant PGL(n+1, \mathbb{C})$ $g = [M]$ $M \in GL(n+1, \mathbb{C})$
  $x \in \mathbb{C}^{n+1} \setminus \{0\}$, $g \cdot [x] = [x]$ ↯
- [Chen–Ogiue, '73]:
  If (X,ω) is KE_μ: $\operatorname{Ric}\omega = \mu\,\omega$ $\mu \in \mathbb{R}$.
  Then,
  $$\left(2(n+1)c_2(X) - nc_1(X)^2\right)\cdot[\omega]^{n-2} \geq 0 \quad \text{Miyaoka–Yau inequality.}$$
  It equals $0 \iff (X,\omega)$ has constant hol. sect. curvature.
- We can construct on X a v.b "canonical extension"
  $$0 \to \mathcal{O}_X \to E_X \to T_X \to 0.$$

S, Q v.b. $\quad 0 \to S \to E \to Q \to 0 \quad H^1(X, \operatorname{Hom}(Q,S))$
$Q = T_X$, $S = \mathcal{O}_X$ $\quad \operatorname{Hom}(Q,S) = \Omega^1_X$, $c_1(X) \in H^1(X, \Omega^1_X)$.

- [Tian, '93]: (X,ω) $KE_1 \Rightarrow E_X$ is polystable w.r.t $c_1(X)$.

Thm: (X,ω) compact Kähler manifold Fano s.t
(SCE): E_X ~~polystable~~ semistable w.r.t $c_1(X)$
(MY): $\left(2(n+1)c_2(X) - nc_1(X)^2\right)\cdot c_1(X)^{n-2} = 0$
$\Rightarrow (X,\omega) \simeq (\mathbb{P}^n, \omega_{FS})$. $\quad X \simeq \mathbb{P}^n$

Proof: $F := \operatorname{End}(E_X)$. (SCE) $\Rightarrow F$ is s.s. wrt. $c_1(X)$.
(MY) $\Rightarrow ch_1(F)\cdot c_1(X)^{n-1} = ch_2(F)\cdot c_1(X)^{n-2} = 0$

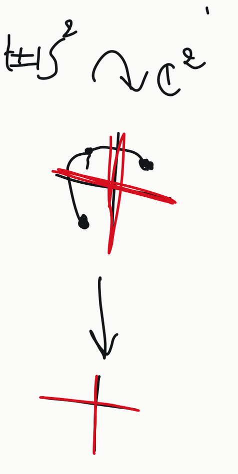
[Simpson] $\Rightarrow F$ is flat
$\Rightarrow E_X$ is proj. flat. $\exists L \in \operatorname{Pic}(X)$,
$\underset{X \text{ Fano}}{\Rightarrow}$ $\mathbb{P}(E_X) \simeq X \times \mathbb{P}^r \Rightarrow E_X = L^{\oplus n+1}$.

$c_1(E_X) = c_1(X) = (n+1)c_1(L)$. $\quad L$ ample.
$\underset{[\text{Kob-Ochiai}]}{\Rightarrow}$ $X \simeq \mathbb{P}^n$.

Y smooth mfd.
$G \curvearrowright Y$ discrete properly, freely.
① $X = Y/G$ can be singular
② $\pi : Y \longrightarrow Y/G$ ramified

Ex: $\{\pm 1\} \curvearrowright \mathbb{C}^2$
$(-1)\cdot(x,y) = (-x,-y)$.
$\mathbb{C}^2/\pm \simeq \ \subseteq \mathbb{C}^3$ $= (w^2 - uv = 0)$
$(x,y) \mapsto (\underset{u}{x^2}, \underset{v}{y^2}, \underset{w}{xy})$

$\{\pm 1\}^2 \curvearrowright \mathbb{C}^2$

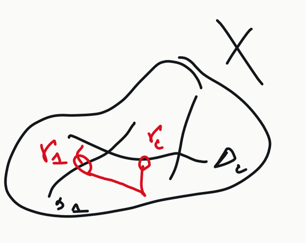


X normal variety
Δ ~~divisor~~ encode branching points in codim. 1.

Df: (X, Δ) $\quad X$ normal variety.
$\Delta = \sum\left(1 - \frac{1}{m_i}\right)\Delta_i$ $\quad \mathbb{Q}$-Weil divisor

(X,Δ) log Fano if (X,Δ) klt
$+$ $-(K_X + \Delta)$ ample.

$\pi_1^{orb}(X,\Delta) = \pi_1(X_{reg} \setminus |\Delta|) / \langle\langle \gamma_i^{m_i} \rangle\rangle$.

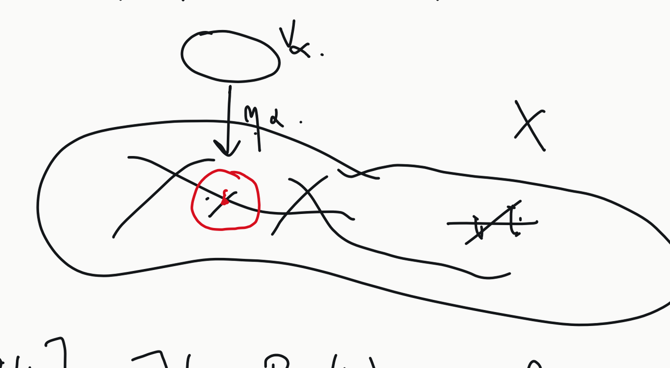


[Claudon–Graf–Guenancia, 24]:
$(\tilde{X}_\Delta, \tilde{\Delta})$ universal cover.

Thm [Greb–Kebekus–Peternell, 2022] $(\Delta = 0)$.
X normal proj. variety Fano. klt
(SCE) The canonical extension $\mathcal{E}_X$ is semistable w.r.t $c_1(X)$
$$0 \to \mathcal{O}_X \to \mathcal{E}_X \to \mathcal{T}_X \to 0$$
(MY): $\left(2(n+1)\hat{c}_2(X) - nc_1(X)^2\right)\cdot c_1(X)^{n-2} = 0$.
$\exists G \leqslant PGL(n+1, \mathbb{C})$, $G \curvearrowright \mathbb{P}^n$ freely in codim. 1.
$X \simeq \mathbb{P}^n/G$

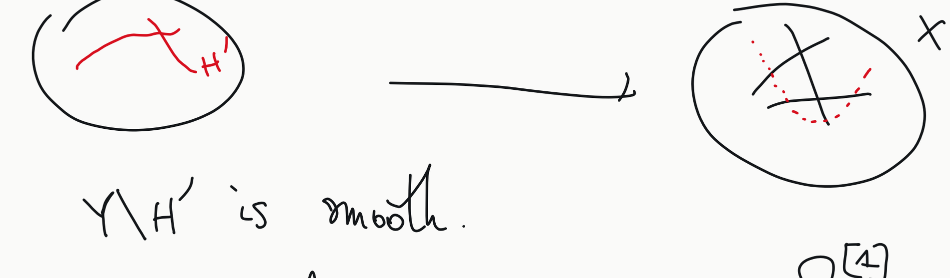
Thm: (X, Δ) log Fano pair. Assume that
(SCE) $\mathcal{E}_{(X,\Delta)}$ is semistable w.r.t. $c_1(X,\Delta) = \frac{1}{m}c_1(-m(K_X + \Delta))$.
(MY): $\left(2(n+1)c_2(X,\Delta) - nc_1(X,\Delta)^2\right)\cdot c_1(X,\Delta)^{n-2} = 0$
Then,
$$(\tilde{X}_\Delta, \tilde{\Delta}) = (\mathbb{P}^n, 0).$$

Idea of the proof: **Step 1**: (X, Δ) is an orbifold.



$\eta_\alpha : V_\alpha \xrightarrow{\text{smooth}} U_\alpha \subseteq X$
Galois finite
$G_\alpha = \operatorname{Gal}(\eta_\alpha)$
$(V_\alpha/G_\alpha, \Delta_{G_\alpha}) \simeq (U_\alpha, \Delta_{|U_\alpha})$

[CGG, '24]: $\exists L \in \operatorname{Pic}(X)$ ample, for $H \in |L|$ general, $f_H : Y \xrightarrow[\text{finite}]{} X$. (normal klt)
$f_H^*\Delta = m_i\Delta_i'$ (red.)
$f^*H = N\,H'$.



$Y \setminus H'$ is smooth.

Adapted canonical extension $\quad \Omega^{[1]}_{X,\Delta,f}$, $\mathcal{T}_{X,\Delta,f}$.
$$0 \to \mathcal{O}_Y \to \mathcal{E}_{X,\Delta,f} \to \mathcal{T}_{X,\Delta,f} \to 0.$$
(SCE): $\exists f_0 : Y \to X$ s.t $\mathcal{E}_{X,\Delta,f_0}$ s.s w.rt $f_0^*c_1(X,\Delta)$ $\Rightarrow \forall f_H$, $\mathcal{E}_{X,\Delta,f_H}$ is s.s. w.r.t $f_H^*c_1(X,\Delta)$
(MY): $\mathcal{F}_H = \operatorname{End}(\mathcal{E}_{X,\Delta,f_H})$ still s.s w.r.t $f_H^*c_1(X,\Delta)$.
$$\left(2(n+1)\hat{c}_2(\mathcal{E}_{X,\Delta,f_H}) - n\hat{c}_1(\mathcal{E}_{X,\Delta,f_H})^2\right)\cdot f_H^*c_1(X,\Delta)^{n-2}$$
$$= (\deg f_H)\left(2(n+1)\hat{c}_2(X,\Delta) - nc_1(X,\Delta)^2\right)\cdot c_1(X,\Delta)^{n-2} = 0$$

$$\hat{ch}_1(\mathcal{F})f_H^*c_1(X,\Delta)^{n-1} = \hat{ch}_2(\mathcal{F})\cdot f_H^*c_1(X,\Delta)^{n-1} = 0.$$
$\Rightarrow \mathcal{F}$ is flat locally free Y.
$$\mathcal{F}_{|U} \simeq \mathcal{O}_U \oplus \mathcal{T}_{X,\Delta,f|U} \oplus \Omega^{[1]}_{X,\Delta,f|U} \oplus \operatorname{End}(\mathcal{T}_{X,\Delta,f|U}).$$